On Semiclassical Equivalence of Green-Schwarz and Pure Spinor Strings in $AdS(5) \times S(5)$

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Abstract

We present a method to study the equivalence at the semiclassical level of the Green-Schwarz and pure spinor formulations of String Theory in $AdS(5) \times S(5)$. This method provides a clear separation of the physical and unphysical sectors of the pure spinor formulation and allows to prove that the two models have not only equal spectra for the fermionic fluctuations but also equal conformal weights (the bosonic ones being equal by construction).

1 Introduction

String theory in $AdS(5) \times S(5)$ is of great relevance being the first and well studied example of AdS/CFT correspondence. The Ramond–Neveu–Schwarz formulation is not suitable to describe string theory in this background, due to the presence of a non vanishing R-R flux. Two approaches are available in this case: the Green-Schwarz (GS) formulation [1] and the pure spinor (PS) one [2]. The GS formulation for Type II superstrings in a general curved background is known for a longtime [3]. That of the PS formulation has been derived in [4] by considering the more general conformal invariant action involving also the momenta $d_{L,\alpha}$, $d_{R,\alpha}$ of the Grassmann–odd superspace coordinates $\theta_{L/R}$ and the ghosts $\lambda^{\alpha}_{L/R}$, $\omega_{L/R,\alpha}$ and by showing that BRS invariance implies the on shell supergravity constraints and holomorphicity of

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BRS currents. For further discussions see [5],[6],[7]. Alternatively, starting from an extended free differential algebra of the superspace geometry [8], [9], [10], one can add to the GS action conformal invariant terms involving $d_{L/R}$ and ghosts to promote the k-symmetry of the GS approach to BRS symmetry [9],[10] following a method proposed in [11] for the heterotic case.

The GS and PS actions in $AdS(5) \times S(5)$ are obtained from the corresponding actions in curved background by setting the background superfields to their $AdS(5) \times S(5)$ values. An incomplete list of papers that discuss the $AdS(5) \times S(5)$ string theory are [12]-[17] in the GS formulation and [18]–[22] in the pure spinor formulation. In the PS case there is a term, proportional to $(d_L M d_R)$ where M is a superfield related to the R-R flux. In $AdS(5) \times S(5)$ M is a constant invertible matrix and, since the other terms are almost linear in $d_{L/R}$, one can integrate over d_L and d_R to get an action that depends on the same superspace variables as in the GS model (in addition to the ghosts). Since $AdS(5) \times S(5)$ is the coset $\frac{PSU(2,2/4)}{SO(4,1) \times SO(5)}$ it is convenient to express the GS and PS actions in terms of the currents, valued in this coset [12]. The bosonic sectors of the GS and PS actions are equal but the fermionic sectors are different. In particular, in the PS formulation the fermionic modes have a second–order kinetic term. Then a natural question to ask is whether the two formulations are equivalent.

It does not seem easy to answer this question in general. However it is possible to study this problem in a simplified set up, that is at the semiclassical level: one expands the GS and PS $AdS(5) \times S(5)$ actions around a classical solution of the bosonic field equations of the string, up to terms quadratic in the fluctuations and then compare their (fermionic) spectra (equivalently their one-loop (fermionic) partition functions). This problem has been faced in two recent papers [23],[24]. In [23] the equality of the fermionic spectra has been proved for a simple family of string motions and in [24] this result has been extended to a generic motion of the string in $AdS(5) \times S(5)$.

In this paper we propose a different method to deal with this problem at the semiclassical level. The method has the advantage that it provides a clear separation of the physical and unphysical sectors of the PS approach and allows to prove not only the equality of the spectra of the physical, fermionic fluctuations of the GS and PS approaches but also their conformal weights. Indeed it is shown that the PS model in the considered approximation contains $8_L + 8_R$ fermions with conformal weight (0,0) and the same (world–sheet (w.s.) dependent) "mass" as in the GS approach, and $2_L + 2_R$

massless fermions with conformal weights (2,0),(-1,0), (0,2),(0,-1) that match the masses and the conformal weights of the b-c ghosts, which are present in the GS approach to fix the w.s. diffeomorphisms in conformal gauge.

2 GS Action in $AdS(5) \times S(5)$

The GS action, in conformal gauge, for a generic curved background is

$$I_{GS} = \int L_{GS} = \frac{R^2}{\pi\alpha'} \int dz^+ dz^- [E_+^a(Z) E_{-,a}(Z) + B_{+,-}(Z)] \tag{1}$$

Where $E^A(Z) \equiv (E^a, E_L^\alpha, E_R^\alpha)$ are the vector-like (E^a) and spinor-like ($E_{L/R}^\alpha$) supervielbeins and B is the NS-NS two-superform. $E_\pm^a$ and $B_{+,-}$ are the pull-back of these forms onto the world sheet with coordinates $z^\pm$ and $Z^M \equiv (x^m, \theta_L^\mu, \theta_R^\mu)$ are the superspace coordinates. If the superspace geometry satisfies the on-shell, supergravity constraints, the action (1) is invariant under k-symmetry,

$$\delta Z^M E_{M,L}^\alpha = (E_-^a \Gamma_a k_L)^\alpha \qquad \delta Z^M E_{M,R}^\alpha = (E_+^a \Gamma_a k_R)^\alpha$$

($k_{L/R}$ being local parameters) that halves the number of fermionic degrees of freedom. In $AdS(5) \times S(5)$ E^a, $a = 0, ...9$ decomposes as $E^a \equiv (E^{\tilde{a}}, E^i)$ where $\tilde{a} = 0, 1, ..4$ refers to $AdS(5)$ and $i = 5, ...9$ refers to $S(5)$. A similar decomposition holds for the Lorentz connection $\omega^{[ab]} \equiv (\omega^{[\tilde{a}\tilde{b}]}, \omega^{[ij]})$ with curvature

$$R^{abcd} = (-R\eta^{\tilde{a}[\tilde{b}}\eta^{tildec]\tilde{d}}, +R\delta^{i[j}\delta^{h],k})) \tag{2}$$

We will assume $R = 1$ in the following. Moreover in this background the only non vanishing component of B is [13] $(E_L B^{(o)} E_R)$ where, with R=1,

$$B^{(o)}_{\alpha,\beta} = (\gamma^{01234})_{\alpha\beta} \equiv (\gamma_*)_{\alpha,\beta} \tag{3}$$

and

$$(\gamma_*)^2 = -1.$$

The matrix γ_* is also equal to the constant matrix $M^{(o)}$ that describes the R-R flux in this background:

$$M^{(o)} = \frac{1}{2 \cdot 5!} F^{abcdf} \gamma_{abcdf} = \gamma_* \tag{4}$$

Then the GS action in $AdS(5) \times S(5)$ is

$$I_{GS} = \int L_{GS} = \int [E^a_+(Z)E_{-,a}(Z) + \frac{1}{2}(E_{L,+}(Z)\gamma_* E_{R,-}(Z)) - \frac{1}{2}(E_{L,-}(Z)\gamma_* E_{R,+}(Z))] \quad (5)$$

Here and in the following $\int ..$ stand for $\int \frac{dz^+ dz^-}{\pi\alpha'}...$

Since $AdS(5) \times S(5)$ is the coset

$$\frac{G}{H} = \frac{PSU(2,2/4)}{SO(4,1) \times SO(5)}$$

it is convenient to express this action in terms of the currents

$$g^{-1}dg = J \equiv J^0 + J^1 + J^2 + J^3 \quad (6)$$

where g is an element of G that transforms under G from the left and under the structure group H on the right and $J = J^A T_A$ is valued in the Lie algebra of G, T_A being the generators of $psu(2,2/4)$, that is $J^2 = J^{2,a}T_a$, $J^1 = J^{1,\alpha}T_{L,\alpha}$ $J^3 = J^{3,\alpha}T_{R,\alpha}$, $J^0 = J^{0,[ab]}T_{[ab]}$. As shown in [13], in terms of these currents the action (5) can be written as

$$I_{GS} = \frac{1}{2}\int str[J^2_+ J^2_- - \frac{1}{2}(J^1_+ J^3_- - J^3_+ J^1_-)] \quad (7)$$

where str denotes the supertrace. This form of the action is useful since it reveals the hidden Z_4 automorphysm of the $AdS(5) \times S(5)$ action, the index r in J^r being its grading under Z_4. The relation with the superspace notations of (1),(5) is: $J^{2,a} = E^a$, $J^{1,\alpha} = E^\alpha_L$, $J^{3,\alpha} = E^\alpha_R$ and $J^{0,[ab]} = \omega^{[ab]}$.

We are interested in studying the motion of a string in $AdS(5) \times S(5)$ at the semiclassical level, i.e. to compute the spectra of the quantum fluctuations around the classical solution. For that one considers the $AdS(5) \times S(5)$ GS action expanded around a classical solution of the bosonic field equations. A generic classical motion of the bosonic string is described by the pull-back of the classical vielbeins e^a_+, e^a_- which in the conformal gauge satisfy the field equations

$$\nabla_+ e^a_- = 0 = \nabla_- e^a_+ \quad (8)$$

and the Virasoro constraints

$$e^a_- e_{-,a} = 0 = e^a_+ e_{+,a} \quad (9)$$

$\nabla_\pm$ are the pull-back of the covariant derivative $\nabla = d + \omega$ where $\omega^{[ab]}$ is the classical Lorentz connection considered previously. It will be convenient to define

$$\not{e}_\pm = e^a_\pm \Gamma_a \equiv e^{\bar{a}}_\pm \Gamma_{\bar{a}} + e^i_\pm \Gamma_i \tag{10}$$

Expanding eq.(5) (or (7)) around this classical solution up to terms quadratic in fluctuations, the action decomposes into a bosonic part and a fermionic part. Since, as already noted and as will be seen below, the bosonic parts are the same for the GS and PS formulations, for the sake of comparison of the two approaches, one can forget the bosonic parts. The fermionic part of the GS lagrangian, expanded up to quadratic terms, is

$$L_{GS} = \frac{1}{2}(\theta_L \not{e}_- \nabla_+ \theta_L) + \frac{1}{2}(\theta_R \not{e}_+ \nabla_- \theta_R) - \frac{1}{2}(\theta_L \not{e}_- \gamma_* \not{e}_+ \theta_R) \tag{11}$$

which is invariant under the simplified k-symmetry

$$\delta\theta_{L/R} = (\not{e}_\mp k_{L/R}).$$

If one define the fermions

$$\theta^{(\pm)}_{PL/R} = P_\pm \theta_{L/R}$$

where $P_\pm$ are the projectors

$$P_\pm = \frac{1}{2(e_+ e_-)} \not{e}_\pm \not{e}_\mp \qquad P_+ + P_- = 1 \tag{12}$$

that project on an 8–dimensional subspace of the 16 dimensional spinorial space, the fermions $\theta^{(\pm)}_{L/R}$ have 8 components each. The k symmetry implies that (36) does not depends on $\theta^{(-)}_L$ and $\theta^{(+)}_R$ so that the lagrangian

$$L_{GS} = \frac{1}{2}(\theta^{+}_L \not{e}_- \nabla_+ \theta^{(+)}_L) + \frac{1}{2}(\theta^{(-)}_R \not{e}_+ \nabla_- \theta^{(-)}_R) - \frac{1}{2}(\theta^{(+)}_L \not{e}_- \gamma_* \not{e}_+ \theta^{(-)}_R) \tag{13}$$

describes 8 left-handed and 8 right-handed massive fermions with conformal weights (0,0), the fermionic partners of the 8 transverse bosons $x^{\hat{m}}$. The values of the fermion masses are determined by the last term of (13). Notice that since this term involves the classical vielbeins $e_\pm(x(z))$, in general the "masses" depend on the w.s. coordinates $z^\pm$.

Let us also recall that fixing the w.s. diffeomorphisms by imposing the conformal gauge requires the GS action to be supplemented with the b-c ghost action

$$\int L_{ghost} = \int [b_{L,--}\partial_+ c_L^- + b_{R,++}\partial_+ c_R^+] \tag{14}$$

where the ghosts $b_{L,--}$, $c_{L,+}$ and $b_{R,++}$, $c_{R,-}$ have conformal weights (2,0), (-1,0) and (0,2), (0,-1) respectively.

3 PS Action in $AdS(5) \times S(5)$

As said before, the pure spinor action in a generic curved background is obtained by adding to the GS action conformal invariant terms involving the fermionic momenta $d_{L/R}$ and the pure–spinor ghosts, so that the PS action becomes invariant under BRS transformations generated by the BRS charge

$$Q_{BRS} = \int dz^-(d_{-,L}\lambda_L) + \int dz^+(d_{+,R}\lambda_R) \tag{15}$$

if the supergravity background is on shell. In $AdS_5 \times S_5$ the PS action takes the form

$$I = \int L_{GS} + \int [(E_{L,+}d_{L,-}) + (E_{R,-}d_{R,+}) - (d_{L,-}\gamma_* d_{R,+})]$$

$$+ I_{\omega-\lambda} \tag{16}$$

where

$$I_{\omega-\lambda} = \int [(\omega_{L-}\nabla_+\lambda_L) + (\omega_{R+}\nabla_-\lambda_R) + R_{abcd}(\omega_{L,-}\Gamma^{a[b}\lambda_L)(\omega_{R,+}\Gamma^{c]d}\lambda_R)] \tag{17}$$

with L_{GS} defined in (5) and R_{abcd} in (2). $\lambda_{L/R}$ are pure spinors satisfying the constraints

$$(\lambda_L\Gamma^a\lambda_L) = 0 = (\lambda_R\Gamma^a\lambda_R)$$

and $I_{\omega-\lambda}$ is invariant under the ω-symmetry

$$\delta\omega_{L/R} = \Lambda^a_{L/R}\Gamma_a\lambda_{L/R}$$

where $\Lambda^a_{L/R}$ are local parameters. Eq. (16) can be integrated over $d_{L/R}$ and expressed in terms of the currents (6), as in the GS case. The result is [18]

$$I_{PS} = \frac{1}{2}\int str[J^2_+ J^2_- + \frac{1}{2}J^1_+ J^3_- + \frac{3}{2}J^3_+ J^1_-] + I_{\omega-\lambda} \tag{18}$$

As anticipated and as it is now clear from (7) and (17), the bosonic sectors in the GS and PS approaches are identical. Therefore, in order to compare the two approaches at the semiclassical level, it is sufficient to consider the fermionic sector of (16) or (18), expanded around the bosonic solution considered in (8), (9), up to terms quadratic in the fluctuations. For the purposes of the present paper it is convenient to start from the non integrated action (16). In this approximation the Lagrangian for the fermionic sector is

$$L = \frac{1}{2}[\theta_L \not{e}_-(\nabla_+\theta_L - \frac{1}{2}\gamma_*\not{e}_+\theta_R) + \theta_R\not{e}_+(\nabla_-\theta_R + \frac{1}{2}\gamma_*\not{e}_-\theta_L)]$$

$$-\,[d_L(\nabla_+\theta_L - \frac{1}{2}\gamma_*\not{e}_+\theta_R) + d_R(\nabla_-\theta_R + \frac{1}{2}\gamma_*\not{e}_-\theta_L) + \frac{1}{2}(d_L\gamma_* d_R)]$$

$$+\,[(\omega_{L-}\nabla_+\lambda_L) + (\omega_{R+}\nabla_-\lambda_R)] \tag{19}$$

Upon integrating (19) over d_L and d_R one gets

$$L = \frac{1}{2}[(\theta_L\not{e}_-(\nabla_+\theta_L - \frac{1}{2}\gamma_*\not{e}_+\theta_R)) + (\theta_R\not{e}_+(\nabla_-\theta_R + \frac{1}{2}\gamma_*\not{e}_-\theta_L))]$$

$$-\,2[(\nabla_+\theta_L - \frac{1}{2}\theta_R\not{e}_+\gamma_*)\gamma_*(\nabla_-\theta_R + \frac{1}{2}\gamma_*\not{e}_-\theta_L))] + [(\omega_{L-}\nabla_+\lambda_L) + (\omega_{R+}\nabla_-\lambda_R)] =$$

$$= -2(\nabla_+\theta_L\gamma_*\nabla_-\theta_R) - \frac{1}{2}(\theta_L\not{e}_-\nabla_+\theta_L) - \frac{1}{2}(\theta_R\not{e}_-\nabla_-\theta_R)$$

$$+\,[(\omega_{L-}\nabla_+\lambda_L) + (\omega_{R+}\nabla_-\lambda_R)] \tag{20}$$

which is the quadratic string action considered in [23], [24].

Let us come back to eq.(19).

Let us define the projectors [2]

$$K_L = \frac{1}{2}(\Gamma^a \gamma_* \lambda_R) \frac{1}{(\lambda_R \gamma_* \lambda_L)} (\lambda_L \Gamma_a) \tag{21}$$

$$K_R = \frac{1}{2}(\Gamma^a \gamma_* \lambda_L) \frac{1}{(\lambda_R \gamma_* \lambda_L)} (\lambda_R \Gamma_a) \tag{22}$$

and $\tilde{K}_L$ and $\tilde{K}_R$ are the transposed of K_L and K_R. Since $Tr K_{L/R} = 5$, $K_{L/R}$ and $1 - K_{L/R}$ decompose the 16-dimensional, spinorial space in 5-dimensional and 11-dimensional subspaces respectively. Notice that

$$K_L \lambda_L = 0 = K_R \lambda_R \tag{23}$$

so that the pure spinors $\lambda_{L/R}$ have 11 components. Moreover

$$\lambda_L \Gamma^a (1 - K_L) = 0 = \lambda_R \Gamma^a (1 - K_R) \tag{24}$$

Other useful identities are

$$\tilde{K}_L + \gamma_* K_R \gamma_* = 0 = \tilde{K}_R + \gamma_* K_L \gamma_* \tag{25}$$

$$(\tilde{K}_L \Gamma^a K_L) = 0 = (\tilde{K}_R \Gamma^a K_R) \tag{26}$$

and, from $(K_L)^2 = K_L$,

$$K_L (\nabla K_L) K_L = 0 = (1 - K_L)(\nabla K_L)(1 - K_L) \tag{27}$$

and the same for K_R. Moreover, by gauge fixing the ω-symmetry , one can impose the conditions

$$\omega_L K_L = 0 = \omega_R K_R \tag{28}$$

In order to avoid problems with the semiclassical approximation it is convenient to assume that $\lambda_{L/R}$ decompose as $\lambda_{L/R} = \lambda_{0,L/R} + \hat{\lambda}_{L/R}$ where

[2] In $AdS(5) \times S(5)$, $(\lambda_R \gamma_* \lambda_L)$ belong to the BRS cohomology and can be assumed to be non vanishing [20].

$\lambda_{0,L/R}$ are classical fields subjects to the field equations $\nabla_{\mp}\lambda_{0,L/R} = 0$ and $\lambda_{0,L/R}$, $\hat{\lambda}_{L/R}$ are pure spinors. Then $K_{L/R}$ become

$$K_L = \frac{1}{2}(\Gamma^a\gamma_*\lambda_{0,R})\frac{1}{(\lambda_{0,R}\gamma_*\lambda_{0,L})}(\lambda_{0,L}\Gamma_a) + O(\hat{\lambda}).$$

Notice that now in (19) (and in (20)) one must add the term

$$-(\omega_{L,-}[R_{abcd}\Gamma^{a[b}\lambda_{0,L})(\lambda_{0,R}\Gamma^{c]d}]\omega_{R,+})$$

coming from the last term of (17). This term does not affects the masslessness of the ghosts $\lambda_{L/R}$ and $\omega_{L/R}$.

The action based on the lagrangian (19) is invariant under the BRS transformations

$$s\nabla_+\theta_L = \nabla_+\lambda_L \qquad s\nabla_-\theta_R = \nabla_-\lambda_R \tag{29}$$

$$s\theta_L\not{e}_- = \lambda_L\not{e}_- \qquad s\theta_R\not{e}_+ = \lambda_R\not{e}_+ \tag{30}$$

$$s\omega_L = -(d_{l,-}(1-K_L)) \qquad s\omega_R = -(d_{R,+}(1-K_R)) \tag{31}$$

$$sd_{L,-} = (\lambda_L\not{e}_-) \equiv (\lambda_L\not{e}_-)K_L \qquad sd_{R,+} = (\lambda_R\not{e}_+) \equiv (\lambda_R\not{e}_+)K_R \tag{32}$$

Notice that we abstain ourselves to define the BRS transformations of $\theta_{L/R}$ themselves i.e. $s\theta_{L/R} = \lambda_{L/R}$ since in a curved background, as $AdS_5\times S_5$, $\lambda^{\alpha}_{L/R}$ transform as spinors under the target–space structure group and $\theta^{\mu}_{L/R}$ transform as odd superspace coordinates. Only the spacetime superfields or forms like $E^A_{L/R}$, B, etc. have definite transformations properties under BRS . In addition notice that, allowing the BRS transformations $s\theta = \lambda$, we would obtain that $s\frac{(\theta_L\gamma_*\lambda_R)}{(\lambda_L\gamma_*\lambda_R)} = 1 = s\frac{(\theta_R\gamma_*\lambda_L)}{(\lambda_L\gamma_*\lambda_R)}$, which trivializes the cohomology. Then we never use them to study the cohomology of our model. But, since in our approximation the model is free, for all other instances the use of $s\theta_{L/R} = \lambda_{L/R}$ is safe.

Then from (29), (30) and (31) it follows that $11_L + 11_R$ components $(1-K_L)\theta_L$ and $(1-K_R)\theta_R$, as well as ω_L and ω_R, are not BRS invariant and therefore the states involving these fields are not present in the physical Fock space which is defined as

$$|\psi> \in \mathcal{F}_{ph} \subset \mathcal{F} \qquad iff \qquad Q_{BRS}|\psi> = 0$$

where $\mathcal{F}$ is the Fock space of the system.The physical space is defined as

$$\mathcal{H}_{ph} = \frac{Ker(Q_{BRS})_{\mathcal{F}}}{Im(Q_{BRS})_{\mathcal{F}}}.$$

Instead

$$\theta_L^{ph} = K_L \theta_L \qquad \theta_R^{ph} = K_R \theta_R \tag{33}$$

are BRS invariant and are physical fields. Notice that, since $e_{\pm}^a$ are classical fields, it follows from (32) that $d_{L,-}K_L$ and $d_{R,+}K_R$ are not invariant under BRS. However if one defines the combinations

$$\hat{d}_{L,-} = d_{L,-} - \theta_L \not{e}_-, \qquad \hat{d}_{L,-} = d_{R,+} - \theta_R \not{e}_+ \tag{34}$$

then

$$d_{L,-}^{ph} = \hat{d}_{L,-} K_L, \qquad d_{R,+}^{ph} = \hat{d}_{R,+} K_R \tag{35}$$

are physical, being BRS invariant but not BRS exact; they can be considered as the conjugate momenta of θ_L^{ph}, θ_R^{ph}.

Here a comment is in order. The appropriate frame to work in the pure spinor approach is the Wick rotated, euclidean version of the theory, with $SO(10)$ as structure group. The pure spinors (as well as any spinor projected with K_L, K_R, $(1-K_L)$, $(1-K_R)$ and any tensor of $SO(10)$) belong to representations of the subgroup $U(5) \subset SO(10)$. Our distinction between physical and un–physical sectors refers to the Euclidean framework. In this case the states involving 10 bosonic fields x^m and the physical fermionic fields $\theta_{L/R}^{ph}$ and $d_{L/R}^{ph}$ have positive norm and $\mathcal{H}_{ph}$ is Hilbert space. When the Wick rotation is reversed to get the Lorentzian version with $SO(1,9)$ as the structure group, $U(5)$ becomes one of its non compact versions and the physical space further reduces to be formed of 8 bosonic and 8 fermionic fields, as will be discussed in the last section of the paper.

The Lagrangian(19) can be obtained as follows: one starts from the Green-Schwarz lagrangian

$$L_{GS} = \frac{1}{2}(\theta_L \not{e}_- \nabla_+ \theta_L) + \frac{1}{2}(\theta_R \not{e}_+ \nabla_- \theta_R) - \frac{1}{2}(\theta_L \not{e}_- \gamma_* \not{e}_+ \theta_R) \tag{36}$$

Notice that its variation is

$$\delta L_{GS} = (\delta\theta_L \not{e}_- [\nabla_+ \theta_L - \frac{1}{2}\gamma_* \not{e}_+ \theta_R]) + (\delta\theta_R \not{e}_+ [\nabla_- \theta_R - \frac{1}{2}\gamma_* \not{e}_- \theta_L])$$

so that L_{GS} is invariant under the κ–symmetry $\delta\theta_L = k_L \not{e}_-$, $\delta\theta_R = k_R \not{e}_+$ as a consequence of the Virasoro constraints $(e^a_- e_{-,a}) = 0 = (e^a_+ e_{+,a})$. Then one adds the new term

$$L_{new} = -(d_{L,-} K_L (\nabla_+ \theta_L - \frac{1}{2}\gamma_* \not{e}_+ \theta_R)) - (d_{R,+} K_R (\nabla_- \theta_R + \frac{1}{2}\gamma_* \not{e}_- \theta_L))$$

$$- \frac{1}{2}(d_{L,-} K_L \gamma_* \tilde{K}_R d_{R,+}) \tag{37}$$

The addition of L_{new} promotes the k-symmetry to a BRS symmetry involving the pure spinors λ_L, λ_R and $L_{GS} + L_{new}$ is invariant under the BRS transformations (29),(30),(31),(32).

The action $\int L$, with L defined in (19), is reached by adding to $\int (L_{GS} + L_{new})$ a suitable, BRS–exact gauge fixing term

$$\int L_{gf} = s \int F_{gf} \tag{38}$$

where F_{gf} is the so called gauge fermion of ghost number $n_{gh} = -1$. By choosing

$$F_{gf} = -(\omega_{L,-} \nabla_+ \theta_L) - (\omega_{R,+} \nabla_- \theta_R) + \frac{1}{2}(\omega_{L,-} \gamma_* \not{e}_+ \theta_R) - \frac{1}{2}(\omega_{R,+} \gamma_* \not{e}_- \theta_L)$$

$$- \frac{1}{4}[\omega_{L,-} \gamma_* (1 - \tilde{K}_R) d_{R,+} - \omega_{R,+} \gamma_* (1 - \tilde{K}_L) d_{L,-}] \tag{39}$$

one gets

$$L_{gf} = sF_{gf} = -(\omega_{L,-} \nabla_+ \lambda_L) - (\omega_{R,+} \nabla_- \lambda_R)$$

$$- (d_{L,-}(1 - K_L) \nabla_+ \theta_L) - (d_{R,+}(1 - K_R) \nabla_- \theta_R)$$

$$\frac{1}{2}(d_{L,-}(1 - K_L) \gamma_* \not{e}_+ \theta_R) - \frac{1}{2}(d_{R,+}(1 - K_R) \gamma_* \not{e}_- \theta_L)$$

$$- \frac{1}{2}(d_{L,-}(1 - K_L) \gamma_* (1 - \tilde{K}_R) d_{R,+}) \tag{40}$$

With this choice any dependence on K_L, K_R disappears from the total action and $L_{GS} + L_{new} + L_{gf}$ coincides with L in (19). Projected with $K_{L/R}$, the first two terms of L_{GS}, neglecting a total derivative, give

$$\frac{1}{2}(\theta_{L/R}\not{e}_{\mp}\nabla_{\pm}\theta_{L/R}) = \theta_L\not{e}_{\mp}K_{L/R}\nabla_{\pm}(K_{L/R}\theta_{L/R})$$

$$+\frac{1}{2}\theta_{L/R}(1-\tilde{K}_{L/R})\not{e}_{\mp}(1-K_{L/R})\nabla_{\pm}((1-K_{L/R})\theta_{L/R}) + L^{(1)}_{L/R} \tag{41}$$

and the last term of L_{GS} yields

$$-\frac{1}{2}(\theta_L\not{e}_{-}\gamma_*\not{e}_{+}\theta_R) = -\frac{1}{2}(\theta_L\tilde{K}_L\not{e}_{-}\gamma_*\not{e}_{+}K_L\theta_R) - \frac{1}{2}(\theta_L\not{e}_{-}K_L\gamma_*\tilde{K}_R\not{e}_{+}\theta_R)$$

$$-\frac{1}{2}(\theta_L(1-\tilde{K}_L)\not{e}_{-}(1-K_L)\gamma_*(1-\tilde{K}_R)\not{e}_{+}(1-K_R)\theta_R) + L^{(2)} \tag{42}$$

where

$$L^{(1)}_{L/R} = (\theta_{L/R}(1-\frac{1}{2}\tilde{K}_{L/R})\not{e}_{\mp}\nabla_{\pm}(K_{L/R})K_{L/R}\theta_{L/R})$$

$$-\frac{1}{2}(\theta_{L/R}(1-\tilde{K}_{L/R})\not{e}_{\mp}\nabla_{\pm}(K_{L/R})(1-K_{L/R})\theta_{L/R}) \tag{43}$$

and

$$L^{(2)} = -\frac{1}{2}(\theta_L\tilde{K}_L\not{e}_{-}\gamma_*\not{e}_{+}(1-K_R)\theta_R) - \frac{1}{2}(\theta_L(1-\tilde{K}_L)\not{e}_{-}\gamma_*\not{e}_{+}K_R\theta_R) \tag{44}$$

Then adding to L_{GS}, the terms L_{new}, given in (37) and L_{gf}, given in (40), and using the definitions (33), (34), (35) as well as

$$\theta^{unph}_L = (1-K_L)\theta_L, \qquad \theta^{unph}_L = (1-K_L)\theta_L \tag{45}$$

$$d^{unph}_{L,-} = [d_{L,-} - \frac{1}{2}\theta^{unph}_L\not{e}_{-}](1-K_L)$$

$$d^{unph}_{R,+} = [d_{R,+} - \frac{1}{2}\theta^{unph}_R\not{e}_{+}](1-K_R) \tag{46}$$

one obtains

$$L_{GS} + L_{new} + L_{gf} = L^{ph} + L^{unph} + L^{(1)} + L^{(2)} +$$

$$- [(d_{L,-}(1-K_L)\gamma_* \not{e}_+ \theta_R) - \frac{1}{2}(d_{R,+}(1-K_R)\gamma_* \not{e}_- \theta_L)]$$

$$+ 2(d_{L,-}(1-K_L)\gamma_*(1-\tilde{K}_R)d_{R,+}) \tag{47}$$

where

$$L^{ph} = -[(d^{ph}_{L,-}\nabla_+\theta^{ph}_L) + (d^{ph}_{R,+}\nabla_-\theta^{ph}_R) + \frac{1}{2}(d^{ph}_{L,-}\gamma_* d^{ph}_{R,+}) +$$

$$\frac{1}{2}(\theta^{ph}_L \not{e}_- \gamma_* \not{e}_+ \theta^{ph}_R)] \tag{48}$$

$$L^{unph} = -(d^{unph}_{L,-}\nabla_+\theta^{unph}_L) - (d^{unph}_{R,+}\nabla_-\theta^{unph}_R)$$

$$- 2(d^{unph}_{L,-}\gamma_* d^{unph}_{R,+}) - (\omega_{L,-}\nabla_+\lambda_L) - (\omega_{R,+}\nabla_-\lambda_R) \tag{49}$$

and

$$L^{(1)} = L^{(1)}_L + L^{(1)}_R \tag{50}$$

where $L^{(1)}_{L/R}$ are defined in (43).

For the purpose of this paper it is not relevant to have an action independent of $K_{L/R}$ and therefore there is a large freedom in the choice of L_{gf}. In other words, adding BRS exact terms to L gives rise to lagrangians, L', equivalent to L. Of course, the terms in the two last rows of (40), (as well as the two last terms in (47)) are, by construction, BRS exact and can be subtracted. (Those of the first two rows of (40) must be retained if one insists to have propagating ghosts $\lambda_{L/R}$ and $\omega_{L/R}$).

Moreover a BRS invariant term which contains $\lambda_L \not{e}_-$ or $\lambda_R \not{e}_+$ is BRS exact being the BRS variation of the same term with $\lambda_{L/R}\not{e}_\mp$ replaced by $d_{L/R}K_{L/R}$. As an example, the first term $L^{(2)}$ in (44) is BRS exact due to the identity

$$(\theta_L \tilde{K}_L \not{e}_- \gamma_* \not{e}_+ (1-K_R)\theta_R) = \frac{(\theta_L\gamma_*\Gamma_a\gamma_*\lambda_L)(\lambda_R\not{e}_+\Gamma^a\not{e}_-(1-K_R)\theta_R)}{2(\lambda_L\gamma_*\lambda_R)} \tag{51}$$

where $\not{v} = v^a \Gamma_a$ and v^a is the one–form with components

$$v^a_\pm \equiv (e^{\tilde{a}}_\pm, -e^i_\pm) \qquad (52)$$

so that

$$\gamma_* \not{e}_\pm = \not{v}_\pm \gamma_*.$$

A similar identity holds for the second term of (44), and therefore $L^{(2)}$ can be subtracted from the action. The same happens for BRS invariant terms that contain a factor $\nabla_\pm \lambda_L$ or $\nabla_\pm \lambda_R$ which are the BRS variation of the same terms with $\nabla_\pm \lambda_L$ replaced by $\nabla_\pm (1 - K_{L/R}) \theta_{L/R}$. In particular terms, like $L^{(1)}_{L/R}$ in (43) (or $L^{(1)}$ in (50)), that contain the factors $\nabla_\pm K_{L/R}$ are BRS exact. However this claim requires a clarification. Indeed it stays on the implicit assumption that $\nabla_\pm K_{L/R}$ produces only terms proportional to $\nabla_\pm \lambda_{L/R}$ i.e. that $s \nabla_\pm \gamma_* = 0$. Since γ_* is the restriction in $AdS_5 \times S_5$ of the Ramond–Ramond flux, which connects left-handed and right-handed spinors and the pure spinor formulation allows for different left-handed and right-handed Lorentz (and Weyl) connections, one might assume that the BRS variation of $\nabla \gamma_*$ vanishes only if the left-handed and right-handed Lorentz connections are equal. However, this also happens if they are different but satisfy a certain condition. If the connections are different

$$\nabla_\pm \gamma_* = \not{\Omega}_{L,\pm} \gamma_* + \gamma_* \not{\Omega}_{R,\pm} = (\not{\Omega}_{L,\pm} - \tilde{\not{\Omega}}_{R,\pm}) \gamma_* = \not{\Omega}_\pm \gamma_* \qquad (53)$$

where $\tilde{\Omega}_{L/R} = \gamma_* \Omega_{L/R} \gamma_*$, $\not{\Omega}_{L/R} = \frac{1}{4} \Omega^{[ab]}_{L/R} \Gamma_{ab}$ are (classical) left-handed and right-handed Lorentz connections and $\not{\Omega} = (\not{\Omega}_L - \tilde{\not{\Omega}}_R)$. If $\not{\Omega} \neq 0$, the second terms of $L^{(1)}_{L/R}$ in (43) are still BRS exact and the first terms are exact provided that

$$\not{e}_\mp \not{\Omega}_\pm = 0. \qquad (54)$$

As we will see, this condition is satisfied by the connection $\not{\Omega}$ that we will need for consistency.

Therefore, neglecting BRS exact terms, the action $\int L$ is equivalent to the action

$$\int L' = \int L^{ph} + \int L^{unph} \qquad (55)$$

L^{ph} describes (in the euclidean) the physical, fermionic, sector and L^{unph} desribes the unphysical sector.

The lagrangian of the physical sector describes a set of 5 + 5 left-handed and 5 + 5 right-handed, physical fields with field equations linear in derivatives, in agreement with the number of the 10 physical, bosonic fields, x^m (5 for(the euclidean version of) AdS_5 and 5 for S_5) with field equations quadratic in derivatives.

As it will be seen in the next section, in the Lorentzian frame this result also agrees with that of the Green-Schwarz formulation, where there are 8 + 8 fermionic, physical fields, and 2 + 2 Grassmann–odd d.o.f. provided by the b–c ghosts.

As for the unphysical sector described by the action $\int L^{unph}$, it follows from eqs. (29), (31) that $\omega_{L,-}$, $\omega_{R,+}$ and θ_L^{unph}, θ_R^{unph} do not belong to the physical Fock space since they are not BRS invariant and λ_L, λ_R and $d_{L,-}(1-K_L)$, $d_{R,+}(1-K_R)$ are swept away by the quotient that defines the physical space $\mathcal{H}_{ph}$, being BRS exact. Moreover it follows from (49) by computing the functional determinant that $d_{L/R}^{unph}$, $\theta_{L/R}^{unph}$, $\lambda_{L/R}$, $\omega_{L/R}$, are massless and form 11 left-handed and 11 right-handed BRS quartets.

4 The Physical Fermionic Sector of the PS Action in $AdS(5) \times S(5)$

Now let us discuss the action $\int L^{ph}$ for the physical sector, in order to clarify the relation of this, free, action and its spectrum with the corresponding action and spectrum of the Green-Schwarz theory. For that comparison we shall work in the Lorentzian frame.

The action $\int L^{ph}$ in (48) is similar to the gauge fixed Green-Schwarz action (13) but with a relevant difference: The GS action describes 4+4 left-handed and 4 + 4 right-handed fermionic fields whereas the physical action $\int L^{ph}$ describes 5+5 left-handed and 5+5 right-handed fermions. Therefore in order to compare them one need 2+2 projectors, let us call them $\Pi^{(4)}_{L/R}$ and $\Pi^{(1)}_{L/R}$ that commute with $K_{L/R}$ and among themselves and project on a 4-dimensional and a 1-dimensional subspace respectively. They satisfy the conditions

$$\Pi^{(4)}_{L/R} + \Pi^{(1)}_{L/R} = K_{L/R} \tag{56}$$

$$\Pi^{(1)}_{L/R}\Pi^{(1)}_{L/R} = \Pi^{(1)}_{L/R}, \qquad \Pi^{(4)}_{L/R}\Pi^{(4)}_{L/R} = \Pi^{(4)}_{L/R} \tag{57}$$

$$\Pi^{(4)}_{L/R}\Pi^{(1)}_{L/R} = 0 = \Pi^{(1)}_{L/R}\Pi^{(4)}_{L/R} \tag{58}$$

Notice however that it is sufficient that these properties hold at the cohomological level i. e. modulo BRS trivial terms. In fact these BRS trivial contributions give rise to terms in the action that are BRS exact and therefore can be subtracted, as discussed before. Therefore in the following we shall imply that the identities considered hold modulo BRS trivial terms.

Let us discuss at first the case when the string motion is restricted to $AdS(5)$. In this case it is not difficult to guess the form of these projectors:

$$\Pi^{(4)}_{L/R} = \frac{1}{2(e_+e_-)} K_{L/R} \not{e}_{\pm} \not{e}_{\mp} K_{L/R} \equiv \frac{1}{2(e_+e_-)} K_{L/R} \not{e}_{\pm} \not{e}_{\mp} \tag{59}$$

$$\Pi^{(1)}_{L/R} = \frac{1}{2(e_+e_-)} K_{L/R} \not{e}_{\mp} \not{e}_{\pm} K_{L/R} = K_{L/R} \frac{(\not{e}_{\mp}\gamma_*\lambda_{R/L})(\lambda_{L/R}\not{e}_{\pm})}{2(e_+e_-)(\lambda_L\gamma_*\lambda_R)} K_{L/R} =$$

$$= \frac{(\not{e}_{\mp}\gamma_*\lambda_{R/L})(\lambda_{L/R}\not{e}_{\pm})}{2(e_+e_-)(\lambda_L\gamma_*\lambda_R)} \tag{60}$$

which indeed satisfy, at the cohomological level, the conditions in (56)- (58). Moreover

$$Tr(\pi^{(4)}_{L/R}) = 4 \qquad Tr(\pi^{(1)}_{L/R}) = 1$$

so that they project on a 4-dimensional and 1-dimensional subspaces.

Now let us come back to the physical action (48) that we write in the form

$$\int L^{ph} = \int [d^{ph}_{L,-} K_L \nabla_+ (K_L\theta_L) + d^{ph}_{R,+} K_R(\nabla_- K_R\theta_R)$$

$$- \frac{1}{2}(d^{ph}_{L,-} K_L \gamma_* \tilde{K}_R d^{ph}_{R,-}) - \frac{1}{2}(\theta_L \tilde{K}_L \not{e}_- \gamma_* \not{e}_+ K_R \theta_R)] \tag{61}$$

and use the identity (56) to replace $K_{L/R}$ with the projectors $\Pi^{(4)}_{L/R}$ and $\Pi^{(1)}_{L/R}$ defined in (59),(60). Then one gets

$$d^{ph}_{L,-}\Pi^{(4)}_L = \frac{1}{2(e_+e_-)}(d^{ph}_{L,-}\not{e}_+)\not{e}_- K_L = \bar{\theta}^{(p)}_L \not{e}_- K_L$$

$$d^{ph}_{L,-}\Pi^{(1)}_L = \frac{1}{2(e_+e_-)(\lambda_L\gamma_*\lambda_R)}(d^{ph}_{L,-}\not{e}_-\gamma_*\lambda_R)\lambda_L\not{e}_+ = \hat{b}_{L,--}\frac{1}{(e_+e_-)}\lambda_L\not{e}_+ \tag{62}$$

$$d^{ph}_{R,+}\Pi^{(4)}_R = \frac{1}{2(e_+e_-)}(d^{ph}_{R,+}\not{e}_-)\not{e}_+K_R = \bar{\theta}^{(p)}_R\not{e}_+K_R$$

$$d^{ph}_{R,+}\Pi^{(1)}_R = \frac{1}{2(e_+e_-)(\lambda_L\gamma_*\lambda_R)}(d^{ph}_{R,+}\not{e}_+\gamma_*\lambda_L)\lambda_R\not{e}_- = \hat{b}_{R,++}\frac{1}{(e_+e_-)}\lambda_R\not{e}_- \tag{63}$$

where we have defined

$$\bar{\theta}^{(p)}_L = \frac{1}{2(e_+, e_-)}d^{ph}_{L,-}\not{e}_+ \qquad \bar{\theta}^{(p)}_R = \frac{1}{2(e_+, e_-)}d^{ph}_{R,+}\not{e}_- \tag{64}$$

and

$$\hat{b}_{L,--} = \frac{1}{2(\lambda_L\gamma_*\lambda_R)}(\hat{d}_{L,-}\not{e}_-\gamma_*\lambda_R) \qquad \hat{b}_{R,++} = \frac{1}{2(\lambda_L\gamma_*\lambda_R)}(\hat{d}_{R,+}\not{e}_+\gamma_*\lambda_L) \tag{65}$$

The definitions (64) look natural since $\bar{\theta}^{(p)}_L$, $\bar{\theta}_R$ (like θ_L,θ_R) have conformal weights (0,0). Moreover we define

$$\Pi^{(4)}_L\theta_L = \theta^{(p)}_L, \qquad \Pi^{(4)}_R\theta_R = \theta^{(p)}_R \tag{66}$$

and

$$\Pi^{(1)}_L\theta_L \equiv \frac{1}{2(e_-e_+)}\frac{(\not{e}_-\gamma_*\lambda_R)}{(\lambda_L\gamma_*\lambda_R)}(\lambda_L\not{e}_+\theta_L) = \frac{(\not{e}_-\gamma_*\lambda_R)}{2(\lambda_L\gamma_*\lambda_R)}\hat{c}^-_L \tag{67}$$

$$\Pi^{(1)}_R\theta_R \equiv \frac{1}{2(e_-e_+)}\frac{(\not{e}_+\gamma_*\lambda_L)(\lambda_R\not{e}_-\theta_R)}{(\lambda_L\gamma_*\lambda_R)} = \frac{(\not{e}_+\gamma_*\lambda_L)}{2(\lambda_L\gamma_*\lambda_R)}\hat{c}^+_R \tag{68}$$

where

$$\hat{c}^-_L = \frac{1}{(e_-e_+)}(\lambda_L\not{e}_+\theta_L) \qquad \hat{c}^+_R = \frac{1}{(e_-e_+)}(\lambda_R\not{e}_-\theta_R) \tag{69}$$

Using the identity (56) the action (61) splits in four parts: $\int L^{ph}_{4,4}$ when both $K_{L/R}$ are replaced by $\Pi^{(4)}_{L/R}$, $\int L_{1,1}$ when both $K_{L/R}$ are replaced by $\Pi^{(1)}_{L/R}$, $\int L_{4,1}$ when the first $K_{L/R}$ is replaced by $\Pi^{(4)}_{L/R}$ and the second one by $\Pi^{(1)}_{L/R}$

and $\int L_{1.4}$ when the first $K_{L/R}$ is replaced by $\Pi^{(1)}_{L/R}$ and the second one by $\Pi^{(4)}_{L/R}$. Then, modulo BRS exact terms, one obtains

$$L^{ph}_{4,4} = (\bar{\theta}^{(p)}_L \not{e}_- \nabla_+ \theta^{(p)}_L) + (\bar{\theta}^{(p)}_R \not{e}_+ \nabla_- \theta^{(p)}_R)$$

$$- \frac{1}{2}(\bar{\theta}^{(p)}_L \not{e}_- \gamma_* \not{e}_+ \bar{\theta}^{(p)}_R) - \frac{1}{2}(\theta^{(p)}_L \not{e}_- \gamma_* \not{e}_+ \theta^{(p)}_R) \tag{70}$$

$$L_{1,1} = [(\hat{b}_{L,--} \nabla_+ \hat{c}^-_L) + (\hat{b}_{R,++} \nabla_- \hat{c}^+_R)] - (\hat{b}_{L,--} \mathcal{G} \hat{b}_{R,++}) \tag{71}$$

where

$$\mathcal{G} = \frac{(\lambda_L \not{e}_+ \gamma_* \not{e}_- \lambda_R)}{2(e_- e_+)^2} \tag{72}$$

and

$$L_{4,1} + L_{1,4} = L^a + L^b + L^c \tag{73}$$

where

$$L^{(a)} = \frac{1}{2(e_- e_+)}[\hat{b}_{L,--}(\lambda_L \not{e}_+ \gamma_* \not{e}_+ \theta^{(p)}_R) + \hat{b}_{R,++}(\lambda_R \not{e}_- \gamma_* \not{e}_- \theta^{(p)}_L)] \tag{74}$$

$$L^{(b)} = \frac{1}{(e_- e_+)}[\hat{b}_{L,--}(\lambda_L \not{e}_+ (\nabla_+ \frac{1}{2(e_- e_+)} \not{e}_+) \not{e}_- \theta^{(p)}_R) +$$

$$\hat{b}_{R,++}(\lambda_R \not{e}_- (\nabla_- \frac{1}{2(e_- e_+)} \not{e}_-) \not{e}_+ \theta^{(p)}_L)] \tag{75}$$

$$L^{(c)} = \frac{1}{(e_- e_+)}[\hat{b}_{L,--}(\lambda_L \not{e}_+ (\nabla_+ K_L) \theta^{(p)}_R) + \hat{b}_{R,++}(\lambda_R \not{e}_- (\nabla_- K_R) \theta^{(p)}_L)] \tag{76}$$

If the string moves only in AdS_5, $\not{e}_\pm$ commute with γ_* so that $\mathcal{G}$ reduces to

$$\mathcal{G} = \frac{(\lambda_L \gamma_* \lambda_R)}{(e_- e_+)} \tag{77}$$

and $L^{(a)}$ in (74) vanishes. As for $L^{(b)}$ and $L^{(c)}$, eq. (75) can be rewritten as

$$L^{(b)} = \frac{1}{(e_-e_+)}[\hat{b}_{L,--}(\lambda_L \not{e}_+(\nabla_+[\frac{e_+^{[a}e_-^{b]}}{(e_-e_+)}]\frac{1}{2}\Gamma_{ab})\theta_R^{(p)})$$

$$+ \hat{b}_{R,++}(\lambda_R \not{e}_-(\nabla_-[\frac{e_-^{[a}e_+^{b]}}{(e_-e_+)}]\frac{1}{2}\Gamma_{ab})\theta_L^{(p)})] \tag{78}$$

and, according to the discussion before Eq.(53), $L^{(c)}$ can be rewritten as

$$L^{(c)} = \frac{1}{(e_-e_+)}[\hat{b}_{L,--}(\lambda_L \not{e}_+ \not{\Omega}_+ \theta_R^{(p)}) + \hat{b}_{R,++}(\lambda_R \not{e}_- \not{\Omega}_- \theta_L^{(p)})] \tag{79}$$

where the Lorentz valued one form $\not{\Omega}$ is the difference between the left-handed and right-handed Lorentz connections. Then, if for consistency we take for $\not{\Omega}_\pm$

$$\not{\Omega}_\pm = -\nabla_\pm[\frac{e_\pm^{[a}e_\mp^{b]}}{(e_-e_+)}]\frac{1}{2}\Gamma_{ab} \tag{80}$$

$L^{(c)}$ and $L^{(b)}$ in (78),(79) cancel and $L_{4,1}^{ph} + L_{1,4}^{ph}$ vanishes at cohomological level. Therefore the action $\int L^{ph}$ is equivalent to the action

$$\int L' = \int L_{4,4}^{ph} + \int L_{1,1}$$

where $L_{4,4}^{ph}$ and $L_{1,1}$ are defined in (70) and (71).

Notice that $\not{\Omega}_\pm$ defined in (89) indeed satisfies the condition (54), as anticipated.

As the Green-Schwarz action, the action $\int L_{4,4}^{ph}$ describes a set of 8 left-handed and 8 right-handed, "massive", fermions with conformal weight (0,0). This action is similar but not identical to the Green-Schwarz action. However these two actions have the same spectrum, as can be seen by showing that their functional determinants are equal. As for $\int L_{1,1}$, the functional determinant of this action does not involves $\mathcal{G}$ being proportional to $\nabla_+\nabla_-\nabla_+\nabla_-$ and therefore $\int L_{1,1}$ describes a massless b-c system.

It is not difficult to modify slightly the procedure to cover the general case, where the string move in all $AdS_5 \times S_5$ [3]. For that purpose let us

[3] Our procedure holds for a generic motion of the string with the exception of those singular motions where $(v_\pm e_\mp) = 0$.

consider the one form v^a defined in (52). Let us recall that

$$\gamma_* \not{e}_\pm = \not{v}_\pm \gamma_* \tag{81}$$

and also notice that

$$(v_\pm e_\mp) = (e_\pm v_\mp) \qquad (v_+ v_+) = 0 = (v_- v_-) \tag{82}$$

Then define the modified propagators

$$\hat{\Pi}^{(4)}_{L/R} = \frac{1}{2(v_\pm e_\mp)} K_{L/R} \not{v}_\pm \not{e}_\mp K_{L/R}$$

$$\hat{\Pi}^{(1)}_{L/R} = \frac{1}{2(v_\pm e_\mp)} (\not{e}_\mp \gamma_* \lambda_{R/L})(\lambda_{L/R} \not{v}_\pm) \tag{83}$$

They satisfy, at the cohomological level, the same conditions (56)-(58) satisfied by $\Pi^{(4)}_{L/R}$ and $\Pi^{(1)}_{L/R}$. Projected with these modified projectors, the analogue of (62), (63), (67), (68) are obtained by replacing in (62),(67) e^a_+ with v^a_+ and in (63),(68) e^a_- with v^a_-. The left-handed fields $\bar{\theta}^{(p)}_L$, $\theta^{(p)}_L$, $\hat{b}_{L,--}$, $\hat{c}^-_L$ and the right-handed fields $\bar{\theta}^{(p)}_R$, $\theta^{(p)}_R$, $\hat{b}_{R,++}$, $\hat{c}^+_R$ are changed accordingly.

For these modified fields we shall maintain the same notations as for the unmodified ones. In particular, $\hat{b}_{L,--}$ and $\hat{b}_{R,++}$ in (65) remain unchanged; $\bar{\theta}^{(p)}_{L/R}$ in(64) become

$$\bar{\theta}^{(p)}_L = \frac{1}{2(v_+, e_-)} d^{ph}_{L,-} \not{v}_+ \qquad \bar{\theta}^{(p)}_R = \frac{1}{2(e_+, v_-)} d^{ph}_{R,+} \not{v}_- \tag{84}$$

and $\hat{c}^\mp_{L/R}$ in (69) become

$$\hat{c}^-_L = \frac{1}{2(e_- v_+)} (\lambda_L \not{v}_+ \theta_L) \qquad \hat{c}^+_R = \frac{1}{2(v_- e_+)} (\lambda_R \not{v}_- \theta_R) \tag{85}$$

As for the action $\int L'$, $L^{ph}_{4,4}$ and $L_{1,1}$ remain unchanged but now $\mathcal{G}$ becomes $\mathcal{G} = \frac{(\lambda_L \not{v}_+ \gamma_* \not{v}_- \lambda_R)}{2(e_- v_+)^2}$ or, using (81) and modulo a BRS trivial term,

$$\mathcal{G} = \frac{(\lambda_L \gamma_* \lambda_R)}{(e_- v_+)}.$$

which agrees with (77) if the string moves only in $AdS(5)$.

Now for $L_{4,1}+L_{1,4}\equiv \hat{L}^{(a)}+\hat{L}^{(b)}+\hat{L}^{(c)}$ one has

$$\hat{L}^{(a)}=\frac{1}{2(e_-v_+)}[\hat{b}_{L,--}(\lambda_L \not{v}_+\gamma_*\not{e}_+\theta_R^{(p)})+\hat{b}_{R,++}(\lambda_R\not{v}_-\gamma_*\not{e}_-\theta_L^{(p)})] \tag{86}$$

$$\hat{L}^{(b)}=\frac{1}{2(e_-v_+)}[\hat{b}_{L,--}(\lambda_L\not{v}_+[(\nabla_+\frac{v_+^a}{(v_+e_-)})e_-^b\Gamma_{ab}]\theta_R^{(p)})$$

$$+\hat{b}_{R,++}(\lambda_R\not{v}_-[(\nabla_-\frac{v_-^a}{(v_-e_+)})e_+^b\Gamma_{ab}]\theta_L^{(p)})] \tag{87}$$

$$\hat{L}^{(c)}=\frac{1}{(v_+e_-)}[\hat{b}_{L,--}(\lambda_L\not{v}_+\hat{\not{\Omega}}_+\theta_R^{(p)})+\hat{b}_{R,++}(\lambda_R\not{v}_-\hat{\not{\Omega}}_-\theta_L^{(p)})] \tag{88}$$

As before $L^{(a)}$ vanishes as a consequence of (81) and (82) and $L^{(b)}$,$L^{(c)}$ cancel each other by choosing for $\hat{\not{\Omega}}$

$$\hat{\not{\Omega}}_\pm=-\nabla_\pm[\frac{v_\pm^{[a}e_\mp^{b]}}{(v_-e_+)}]\frac{1}{2}\Gamma_{ab} \tag{89}$$

which also satisfies the condition (54). Therefore, as before, $L_{4,1}+L_{1,4}$ vanishes at cohomological level.

Then the action $\int L$ is equivalent to the action

$$\int L'=\int L_{4,4}^{ph}+\int L_{1,1}$$

where $L_{4,4}^{ph}$ and $L_{1,1}$ are defined in (70) and (71).

The action $\int L_{4,4}^{ph}$ describes a set of 8 left-handed and 8 right-handed, massive, fermions with conformal weight (0,0) and the same "mass" of the fermions in the GS action. Indeed (70) reduces to eq. (13) with the redefinitions

$$\theta_L^{(+)}=e^{\frac{-i\pi}{4}}\begin{pmatrix}\theta_L^{(1)}\\ \theta_L^{(2)}\end{pmatrix}\qquad,\qquad \theta_R^{(-)}=e^{\frac{i\pi}{4}}\begin{pmatrix}\theta_R^{(1)}\\ \theta_R^{(2)}\end{pmatrix}$$

where

$$\theta_{L/R}^{(1)}=e^{\frac{\pm\pi}{4}}(\theta_{L/R}^{(p)}-\bar{\theta}_{L/R}^{(p)})\qquad,\qquad \theta_{L/R}^{(2)}=e^{\frac{\mp\pi}{4}}(\theta_{L/R}^{(p)}+\bar{\theta}_{L/R}^{(p)})$$

As for the action $\int L_{1,1}$, it describes a left-handed and a right-handed pair of massless, anticommuting scalars with conformal weights (2,0), (-1,0), (0,2) and (0,-1) which are equivalent to the b-c ghost system of the Green-Schwarz approach.

In conclusion we have proved the equivalence at the semiclassical level of the Green-Schwarz and pure spinor formulations of a string moving in $AdS_5 \times S_5$, with a method that allows for a clear separation of the physical and unphysical fermionic sectors, by showing that the quadratic, physical fluctuations have not only the same spectrum but also the same conformal weights.

As a last remark, it is interesting to notice the relation between our fields $\hat{b}_{L/R,\mp\mp}$ and the b-fields [20, 21, 22] for the pure spinor string theory in $AdS_5 \times S_5$. Indeed, in our notations, the $AdS_5 \times S_5$ b–fields are

$$b = \frac{1}{(\lambda_L \gamma_* \lambda_R)}[\frac{1}{2}(E_{R,-}\gamma_* E^a_- \Gamma_a \gamma_* \lambda_R) + \frac{1}{4}(E_{R,-}N^{ab}_{L,-}\Gamma_{ab}\lambda_R) + \frac{1}{4}(E_{R,-}j_{L,-}\lambda_R)] \tag{90}$$

and

$$\bar{b} = \frac{1}{(\lambda_L \gamma_* \lambda_R)}[\frac{1}{2}(E_{L,+}E^a_+ \Gamma_a \gamma_* \lambda_L) + \frac{1}{4}(E_{L,+}N^{ab}_{R,+}\Gamma_{ab}\lambda_L) + \frac{1}{4}(E_{L,+}j_{R,+}\lambda_L)] \tag{91}$$

where $N_{L/R,\mp} = (\omega_{L/R\mp}\Gamma^{ab}\lambda_{L/R})$ and $j_{L/R,\mp} = (\omega_{L/R\mp}\lambda_{L/R})$ and one recovesr our expressions for $b_{L/R,\mp\mp}$ by neglecting the last two terms in (90) and (91) (which are of higher order), replacing $E^a_\pm$ with the classical solutions $e^a_\pm$ and using the fields equations of $d_{L/R}$. Also let us notice that in eq.(3,35) of [20] the expression for the zero-modes c_o and $\bar{c}_o$ of the c-fields in a R-R plane-wave background seems to be related to our fields $\hat{c}^-_L$, $\hat{c}^+_R$ in (69) (or (85)) [4].

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